NORTH-HOLLAND

Eigenvalues of Matrices With Given Block Upper Triangular Part

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ABSTRACT

We describe the possible eigenvalues of 2×2 block matrices M_X of the form $M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix}$, where A, B, and C are given matrices and X can be any matrix.

1. INTRODUCTION

Let $\mathbf{C}^{p \times q}$ denote the space of all $p \times q$ complex matrices. Let $A \in \mathbf{C}^{n \times n}$, $B \in \mathbf{C}^{m \times m}$, and $C \in \mathbf{C}^{n \times m}$ be given, and let $\lambda_1, \dots, \lambda_{n+m}$ be $n + m$ (not necessarily different) complex numbers. In this note we give necessary and sufficient conditions for the existence of $X \in \mathbf{C}^{m \times n}$ for which $\lambda_1, \dots, \lambda_{n+m}$ are the eigenvalues of the $(n + m) \times (n + m)$ matrix M_X $[= M_X(A, B, C)]$ defined by

$$M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix}.$$

In the case when $n = 1$ or $m = 1$, such conditions were given by Gohberg and Gu [4]. See also [1] and [2] for related results. Clearly, if $\lambda_1, \dots, \lambda_{n+m}$ are the eigenvalues of M_X for some X, then

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$$\sum_{i=1}^{n+m} \lambda_i = \operatorname{tr} A + \operatorname{tr} B.$$

Theorem 1 below gives the case when this necessary condition is also sufficient.

A pair (S, R) of matrices $S \in \mathbf{C}^{n\times n}$ and $R \in \mathbf{C}^{n\times m}$ is called full range (or controllable) if $\sum_{i=0}^{n-1} \operatorname{Im}(S^i R) = \mathbf{C}^n$. A classical theorem of Wonham [6] (see also [5, p. 203]) asserts that the pair (S, R) is full range if and only if for each n-tuple $\{\lambda_1, \dots, \lambda_n\}$ of complex numbers there exists F such that the eigenvalues of $S + RF$ consists exactly of $\lambda_1, \dots, \lambda_n$. Our argument in this note also gives conditions for an $(n+m)$-tuple $\{\lambda_1, \dots, \lambda_{n+m}\}$ of complex numbers to exist F such that

$$\sigma(A + CF) = \{\lambda_1, \dots, \lambda_n\} \quad \text{and} \quad \sigma(B - FC) = \{\lambda_{n+1}, \dots, \lambda_{n+m}\},$$

where A, B, and C are matrices given as above.

2. THE FULL RANGE CASE

In this section we consider the case when the pairs (A, C) and (B^T, C^T) are full range. (For a matrix N, N^T denotes the transposed matrix of N.) Let Z_n denote the set of all n-tuples $\{\lambda_1, \dots, \lambda_n\}$ of complex numbers (with repetition allowed). The main result is the following.

THEOREM 1. *Let $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{m\times m}$, and $C \in \mathbf{C}^{n\times m}$ be matrices such that the pairs (A, C) and (B^T, C^T) are full range, and assume that $AC + CB$ is not a scalar multiple of C. Then, for $\{\lambda_1, \dots, \lambda_{n+m}\} \in Z_{n+m}$, there exists $X \in \mathbf{C}^{m\times n}$ such that $\lambda_1, \dots, \lambda_{n+m}$ are the eigenvalues of M_X if and only if*

$$\sum_{i=1}^{n+m} \lambda_i = \operatorname{tr} A + \operatorname{tr} B. \tag{1}$$

Note that if (A, C) and (B^T, C^T) are full range and $AC + CB$ is a scalar multiple of C, i.e., $AC + CB = dC$ for some complex number d, then C is invertible. So Theorem 1 can be applied to a triplet (A, B, C) such that

(A, C) and (B^T, C^T) are full range and C is not invertible, in particular $n \neq m$.

For the case when $AC + CB$ is a scalar multiple of C, we have the following proposition.

PROPOSITION 2. *Let A, B, and C be $n \times n$ matrices satisfying $AC + CB = dC$ for some complex number d, and suppose that C is invertible. Then, for $\{\lambda_1, \dots, \lambda_{2n}\} \in Z_{2n}$, there exists $X \in \mathbf{C}^{n \times n}$ such that $\lambda_1, \dots, \lambda_{2n}$ are the eigenvalues of M_X if and only if there exists a permutation τ on $\{1, 2, \dots, 2n\}$ such that*

$$\lambda_{\tau(i)} + \lambda_{\tau(n+i)} = d \qquad (i = 1, 2, \dots, n). \tag{2}$$

Proof. For each $X \in \mathbf{C}^{n \times n}$, simple calculations show

$$\begin{bmatrix} C & 0 \\ B & I \end{bmatrix}^{-1} M_X \begin{bmatrix} C & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} C^{-1}AC + B & I \\ Y & 0 \end{bmatrix} = \begin{bmatrix} dI & I \\ Y & 0 \end{bmatrix},$$

where $Y = XC - BC^{-1}AC$, and therefore we have

$$\begin{aligned} \det(\lambda I - M_X) &= \det \begin{bmatrix} (\lambda - d)I & -I \\ -Y & \lambda I \end{bmatrix} \\ &= \det\left[(\lambda^2 - d\lambda)I - Y\right] \\ &= \prod_{i=1}^{n} (\lambda^2 - d\lambda - \mu_i), \end{aligned} \tag{3}$$

where $\mu_1, \dots, \mu_n$ are the eigenvalues of Y. It follows that the eigenvalues $\lambda_1, \dots, \lambda_{2n}$ of M_X satisfy (2). Conversely, suppose that $2n$ numbers $\lambda_1, \dots, \lambda_{2n}$ satisfy (2). Let

$$Y = -\operatorname{diag}\left(\lambda_{\tau(1)}\lambda_{\tau(n+1)}, \lambda_{\tau(2)}\lambda_{\tau(n+2)}, \dots, \lambda_{\tau(n)}\lambda_{\tau(2n)}\right)$$

and

$$X = (Y + BC^{-1}AC)C^{-1}.$$

Then since $Y = XC - BC^{-1}AC$, (2) and (3) show that $\lambda_1, \dots, \lambda_{2n}$ are the eigenvalues of M_X. ∎

In order to prove Theorem 1 we start with some lemmas.

LEMMA 3. *Let $S \in \mathbf{C}^{n\times n}$ and $R \in \mathbf{C}^{m\times n}$. If $R \neq 0$ and $m \geqslant 2$, then there exists $F \in \mathbf{C}^{n\times m}$ such that $S + FR$ has the same characteristic polynomial as S and RF is not diagonal.*

Proof. Take an invertible matrix P such that $P^{-1}SP$ is upper triangular. Let l be an integer such that the lth column of $RP = (r_{ij})\ (\neq 0)$ is not zero, but its jth column is zero for all $j < l$, and suppose that $r_{kl} \neq 0$. Let $G = (g_{ij})$ be an $n \times m$ matrix whose lth row has entries

$$g_{lj} = 1 \quad \text{for all } j \neq k, \qquad g_{lk} = -r_{kl}^{-1} \sum_{i\neq k} r_{il}$$

and all other rows are zero. Then $P^{-1}SP + GRP$ is upper triangular and has the same diagonals as $P^{-1}SP$. Therefore the characteristic polynomial of $S + PGR$ is equal to that of S. Also, all entries in the kth row of RPG are equal to nonzero r_{kl} except for its (k, k) entry. Thus the matrix $F = PG$ satisfies the required conditions. ∎

LEMMA 4. *Suppose that $S \in \mathbf{C}^{n\times n}$ is not scalar. Then, for any $\{\lambda_1, \dots, \lambda_{2n}\} \in Z_{2n}$ with $\sum_{i=1}^{2n} \lambda_i = \operatorname{tr} S$, there exists a matrix F such that*

$$\sigma(S - F) = \{\lambda_1, \dots, \lambda_n\} \quad \textit{and} \quad \sigma(F) = \{\lambda_{n+1}, \dots, \lambda_{2n}\}. \tag{4}$$

Proof. By a result in [3] there exists an invertible matrix P such that $P^{-1}SP = (s_{ij})$ has diagonal entries $s_{ii} = \lambda_i + \lambda_{n+i}$ $(i = 1, 2, \dots, n)$. Let $G = (g_{ij})$ be the upper triangular matrix defined by

$$g_{ii} = \lambda_{n+i}, \qquad g_{ij} = s_{ij} \quad (i < j), \quad \text{and} \quad g_{ij} = 0 \quad (i > j).$$

Then $F = PGP^{-1}$ satisfies (4). ∎

Proof of Theorem 1. It is clear that the condition (1) is necessary. Conversely, assume that $\{\lambda_1, \dots, \lambda_{n+m}\}$ satisfies (1). For $F \in \mathbf{C}^{m\times n}$ and

$X \in \mathbf{C}^{m \times n}$, we have

$$\begin{bmatrix} I & 0 \\ F & I \end{bmatrix}^{-1} \begin{bmatrix} A & C \\ X & B \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} = \begin{bmatrix} A + CF & C \\ X + BF - FA - FCF & B - FC \end{bmatrix},$$

so if $X = FCF + FA - BF$, then

$$\det(\lambda I - M_X) = \det[\lambda I - (A + CF)] \det[\lambda I - (B - FC)].$$

Thus it suffices to show that there exists $F \in \mathbf{C}^{m \times n}$ such that

$$\sigma(A + CF) = \{\lambda_1, \dots, \lambda_n\} \quad \text{and} \quad \sigma(B - FC) = \{\lambda_{n+1}, \dots, \lambda_{n+m}\}. \tag{5}$$

First we consider the case when C is invertible and so $n = m$. In this case, by assumption $C^{-1}AC + B$ is not scalar and $\operatorname{tr}(C^{-1}AC + B) = \sum_{i=1}^{2n} \lambda_i$. Hence it follows from Lemma 4 that there exists a matrix G such that

$$\sigma(C^{-1}AC + B - G) = \{\lambda_1, \dots, \lambda_n\} \quad \text{and} \quad \sigma(G) = \{\lambda_{n+1}, \dots, \lambda_{2n}\}.$$

Then the matrix $F = (B - G)C^{-1}$ satisfies (5).

Next assume that C is not invertible and $n \leqslant m$. (The case when $n > m$ can be similarly proved by considering the transposed matrices of A, B, and C.) Let $r = \operatorname{rank} C$ ($\geqslant 1$), and take invertible matrices P and Q such that

$$P^{-1}CQ = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \tag{6}$$

where I_r is the $r \times r$ identity matrix. We write

$$P^{-1}AP = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad Q^{-1}BQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{7}$$

where the matrices A_{11} and B_{11} are of sizes $(n - r) \times (n - r)$ and $(m - r) \times (m - r)$, respectively. Since the pair $(P^{-1}AP, P^{-1}C)$ together with (A, C) is full range, it follows that (A_{11}, A_{12}) is full range too. Similarly, (B_{11}^T, B_{21}^T) is full range. Hence, by the eigenvalue assignment theorem of Wonham (see [6] or [5, p. 203]) there exist $G \in \mathbf{C}^{r \times (n-r)}$ and $H \in \mathbf{C}^{(m-r) \times r}$ such that

$$\sigma(A_{11} + A_{12}G) = \{\lambda_{r+1}, \dots, \lambda_n\} \tag{8}$$

and

$$\sigma(B_{11} - HB_{21}) = \{\lambda_{n+r+1}, \dots, \lambda_{n+m}\}. \tag{9}$$

The conditions (1), (8), and (9) imply

$$\operatorname{tr}(A_{22} - GA_{12}) + \operatorname{tr}(B_{22} + B_{21}H) = \sum_{i=1}^{r} (\lambda_i + \lambda_{n+i}).$$

Moreover, in case $r \geqslant 2$, we can choose G and H in such a way that

$$S = (A_{22} - GA_{12}) + (B_{22} + B_{21}H)$$

is not scalar. Indeed, since (B_{11}^T, B_{21}^T) is full range, $B_{21} \neq 0$ (note that $m - r$ is nonzero, since C is not invertible and $m \geqslant n$ by our assumption), so by Lemma 3 there exists $H' \in \mathbf{C}^{(m-r)\times r}$ such that $B_{11} - (H + H')B_{21}$ has the same characteristic polynomial as $B_{11} - HB_{21}$ and $B_{21}H'$ is not diagonal. Therefore, if S is scalar, the matrix S defined by using $H + H'$ instead of H is not scalar and we can replace H by $H + H'$. Thus, in case $r \geqslant 2$, we can apply Lemma 4 to obtain a matrix $D \in \mathbf{C}^{r\times r}$ such that

$$\sigma(A_{22} - GA_{12} - D) = \{\lambda_1, \dots, \lambda_r\} \tag{10}$$

and

$$\sigma(B_{22} + B_{21}H + D) = \{\lambda_{n+1}, \dots, \lambda_{n+r}\}. \tag{11}$$

In case $r = 1$, we take $D = A_{22} - GA_{12} - \lambda_1$, which satisfies (10) and (11).

Let us consider

$$F = Q\begin{bmatrix} 0 & L \\ K & -D \end{bmatrix}P^{-1}, \tag{12}$$

where K and L are matrices of sizes $r \times (n - r)$ and $(m - r) \times r$, respectively. By simple calculations using (6) and (7), we have

$$\begin{bmatrix} I & 0 \\ G & I \end{bmatrix}^{-1} P^{-1}(A + CF)P\begin{bmatrix} I & 0 \\ G & I \end{bmatrix} = \begin{bmatrix} A_{11} + A_{12}G & A_{12} \\ V + K - DG & A_{22} - GA_{12} - D \end{bmatrix}$$

and

$$\begin{bmatrix} I & H \\ 0 & I \end{bmatrix}^{-1} Q^{-1}(B - FC)Q\begin{bmatrix} I & H \\ 0 & I \end{bmatrix} = \begin{bmatrix} B_{11} - HB_{21} & W - L - HD \\ B_{21} & B_{22} + B_{21}H + D \end{bmatrix},$$

where

$$V = A_{21} + A_{22}G - GA_{11} - GA_{12}G$$

and

$$W = B_{12} + B_{11}H - HB_{22} - HB_{21}H.$$

Therefore, it follows from (8), (9), (10), and (11) that the matrix F of (12) with $K = DG - V$ and $L = W - HD$ satisfies (5), and the proof is complete. ■

The following result shown in the proof of Theorem 1 is of some interest in connection with the eigenvalue assignment theorem of Wonham [6].

COROLLARY 5. *Let $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{m\times m}$, and $C \in \mathbf{C}^{n\times m}$ be matrices such that (A, C) and (B^T, C^T) are full range and $AC + CB$ is not a scalar multiple of C. Let $\{\alpha_1, \dots, \alpha_n\} \in Z_n$ and $\{\beta_1, \dots, \beta_m\} \in Z_m$. Then there exists $F \in \mathbf{C}^{m\times n}$ such that*

$$\sigma(A + CF) = \{\alpha_1, \dots, \alpha_n\} \quad \textit{and} \quad \sigma(B - FC) = \{\beta_1, \dots, \beta_m\}$$

if and only if

$$\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{m} \beta_i = \operatorname{tr} A + \operatorname{tr} B.$$

3. THE GENERAL CASE

We now consider the general case. Let $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{m\times m}$, and $C \in \mathbf{C}^{n\times m}$ be given. As is well known in systems theory, there are invertible matrices P and Q such that

$$P^{-1}AP = \begin{bmatrix} A_2 & 0 \\ A_{12} & A_1 \end{bmatrix}, \qquad Q^{-1}BQ = \begin{bmatrix} B_1 & 0 \\ B_{21} & B_2 \end{bmatrix}, \tag{13}$$

and

$$P^{-1}CQ = \begin{bmatrix} 0 & 0 \\ C_1 & 0 \end{bmatrix}, \tag{14}$$

where (A_1, C_1) and (B_1^T, C_1^T) are full range pairs (see [5, p. 82]). Thus, for each $X \in \mathbf{C}^{m \times n}$, the matrix M_X is similar to

$$\begin{bmatrix} A_2 & 0 & 0 & 0 \\ A_{12} & A_1 & C_1 & 0 \\ Y_{12} & Y_1 & B_1 & 0 \\ Y_2 & Y_{21} & B_{21} & B_2 \end{bmatrix},$$

where

$$Q^{-1}XP = \begin{bmatrix} Y_{12} & Y_1 \\ Y_2 & Y_{21} \end{bmatrix},$$

so that the characteristic polynomial of M_X can be written as

$$\det(\lambda I - M_X) = \det(\lambda I - A_2)\det(\lambda I - B_2)\det\left(\lambda I - \begin{bmatrix} A_1 & C_1 \\ Y_1 & B_1 \end{bmatrix}\right). \tag{15}$$

Moreover, using the Smith form of the matrix polynomial $(\lambda I - A, C)$ (see [5, p. 653]), we see that the characteristic polynomial of A_2 is the product of the invariant polynomials of $(\lambda I - A, C)$, in other words, it is a greatest common divisor of all minors of order n of $(\lambda I - A, C)$. Similarly, the characteristic polynomial of B_2 is a greatest common divisor of all minors of order m of $(\lambda I - B^T, C^T)$.

THEOREM 6. *Let* $A \in \mathbf{C}^{n \times n}$, $B \in \mathbf{C}^{m \times m}$, $C \in \mathbf{C}^{n \times m}$, *and* $\{\lambda_1, \dots, \lambda_{n+m}\} \in Z_{n+m}$. *Let* $\alpha(\lambda)$ *(respectively,* $\beta(\lambda)$*) be the (monic) greatest common divisor of all minors of order n of* $(\lambda I - A, C)$ *(respectively, of* *order m of $(\lambda I - B^T, C^T)$), and let*

$$\alpha(\lambda)\beta(\lambda) = \prod_{i=1}^{s}(\lambda - \mu_i).$$

(a) *When $AC + CB$ is not a scalar multiple of C, there exists $X \in \mathbf{C}^{m\times n}$ such that $\lambda_1, \dots, \lambda_{n+m}$ are the eigenvalues of M_X if and only if there exists a permutation τ on $\{1, 2, \dots, n+m\}$ such that*

$$\lambda_{\tau(i)} = \mu_i \quad (i = 1, 2, \dots, s) \quad \text{and} \quad \sum_{i=s+1}^{n+m} \lambda_{\tau(i)} = \operatorname{tr} A + \operatorname{tr} B - \sum_{i=1}^{s} \mu_i.$$

(b) *When $AC + CB = dC$ for some scalar d (in this case we have $n + m - s = 2\operatorname{rank} C$), there exists $X \in \mathbf{C}^{m\times n}$ such that $\lambda_1, \dots, \lambda_{n+m}$ are the eigenvalues of M_X if and only if there exists a permutation τ on $\{1, 2, \dots, n+m\}$ such that*

$$\lambda_{\tau(i)} = \mu_i \quad (i = 1, 2, \dots, s) \quad \text{and}$$

$$\lambda_{\tau(s+i)} + \lambda_{\tau(s+r+i)} = d \qquad (i = 1, 2, \dots, r),$$

where $r = \operatorname{rank} C$.

Proof. Let A_i, B_i $(i = 1, 2)$ and C_1 be the matrices in (13) and (14). As remarked above, $\alpha(\lambda)$ and $\beta(\lambda)$ are the characteristic polynomials of A_2 and B_2, respectively, which implies $\operatorname{tr} A_1 + \operatorname{tr} B_1 = \operatorname{tr} A + \operatorname{tr} B - \sum_{i=1}^{s} \mu_i$. It is easy to see that for a complex number d, the equality $AC + CB = dC$ is equivalent to $A_1C_1 + C_1B_1 = dC_1$. Moreover, if $A_1C_1 + C_1B_1 = dC_1$ for some d, then since (A_1, C_1) and (B_1^T, C_1^T) are full range, C_1 is invertible and therefore we have $n + m - s = 2\operatorname{rank} C_1 = 2r$. Thus the theorem follows from (15), Theorem 1, and Proposition 2. ∎

REFERENCES

1 J. A. Ball, I. Gohberg, L. Rodman, and T. Shalom, On the eigenvalues of matrices with given upper triangular part, *Integral Equations Operator Theory* 13:488–497 (1990).

2 H. Du and C. Gu, On the spectra of operator completion problems, in *New Aspects in Interpolation and Completion Theories* (I. Gohberg, Ed.), Birkhäuser, Basel, 1993, pp. 103–117.

3 P. A. Fillmore, On similarity and the diagonal of a matrix, *Amer. Math. Monthly* 76:167–169(1969).
4 I. Gohberg and C. Gu, On a completion problem for matrices, in *New Aspects in Interpolation and Completion Theories* (I. Gohberg, Ed.), Birkhäuser, Basel, 1993, pp. 203–217.
5 I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, New York, 1986.
6 W. M. Wonham, On pole assignment in multi-input controllable linear systems, *IEEE Trans. Automat. Control* 12:660–665 (1967).

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